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An Ad Hoc SOR Method*

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An ad hoc method is presented for determining a sequence of successive overrelaxation factors (ω) that, although having no rigorous foundations at the moment, appears to produce a highly competitive iteration scheme for several classes of difference equations. A different ω is chosen for each difference equation.

INTRODUCTION

In using the successive overrelaxation method (SOR) to solve a five-point difference equation (indeed, to solve any linear system), one needs an estimate of the optimal relaxation factor. This can be obtained by estimating the eigenvalues of the corresponding Jacobi method matrix. For simple equations and regions, a good estimate may be available; however, for nonseparable and/or nonsymmetric linear systems, this may not be so. Currently, methods are being developed to determine the relaxation factor dynamically (see, e.g., [6]). Some earlier approaches are described in [7] and references cited therein.

Another approach has been to use a different relaxation factor at each point (see, e.g., [1, 9, 10]). These methods have frequently been designed for matrices with certain characteristic properties and thus may not be effective in general. We propose here a general method for determining a different relaxation factor for each point, depending on the coefficients of the difference equations and the nature of the region and boundary conditions apparently subject only to the condition that SOR converges. Our method is similar in some respects to that of Brazier [1]. Unfortunately, we also have no analytic results. We tout the method on the basis of its behavior in a limited number of problems.

In Section 1 we describe our approach while in Sections 2 and 3 we describe modifications of the basic procedure due to boundary conditions and region shape, respectively. Section 4 contains a statement of the problems we applied the method to and a discussion of the results. In Section 5 we note other variable relaxation factor techniques and in Section 6, conclusions.

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1. The Basic Procedure

Consider the five-point difference equation

$$a_1 x_{i+1,j} + a_2 x_{i,j+1} + a_3 x_{i-1,j} + a_4 x_{i,j-1} - a_0 x_{ij} = b_{ij}, \tag{1.1}$$

where the a_k are, in general, functions of i and j over some grid. We shall write the above in stencil notation as

$$\begin{bmatrix} & a_2 & \\ a_3 & -a_0 & a_1 \\ & a_4 & \end{bmatrix} x = b. \tag{1.2}$$

Our aim here is to solve (1.1) by an ad hoc SOR method using a different relaxation factor at each point.

Consider stencil (1.2) at a particular point in the grid. Assuming, for the moment, that the difference equation has constant coefficients throughout the region, we determine the optimal relaxation factor and apply it to the point in question. This procedure is repeated at each point during the first iteration. We can then save the relaxation factors for the subsequent iterations, or subsequent solutions.

Considering (1.1) with constant coefficients, the corresponding Jacobi stencil is

$$\begin{bmatrix} & \dfrac{a_2}{a_0} & \\ \dfrac{a_3}{a_0} & 0 & \dfrac{a_1}{a_0} \\ & \dfrac{a_4}{a_0} & \end{bmatrix}. \tag{1.3}$$

Since the coefficients are constant, this equation is separable. To determine the eigenvalues of the Jacobi matrix, we are led to solve the eigenvalue difference equations

$$\begin{array}{lll} \text{(a)} & a_3 v_{i-1} + a_1 v_{i+1} = \mu_1 a_0 v_i, & v_0 = v_{N+1} = 0, \\ \text{(b)} & a_4 w_{j-1} + a_2 w_{j+1} = \mu_2 a_0 w_i, & w_0 = w_{M+1} = 0, \end{array} \tag{1.4}$$

where we have $N \times M$ unknowns in the region and where we have assumed given boundary conditions for the original problem. This leads to the Jacobi eigenvalues

$$\mu_{p,q} = \mu_1 + \mu_2 = \frac{2}{a_0}\left[\sqrt{a_1 a_3}\cos\frac{\pi p}{N+1} + \sqrt{a_2 a_4}\cos\frac{\pi q}{M+1}\right]. \tag{1.5}$$

Since the matrix is obtained from a five-point stencil using the "natural ordering," it is known that for each μ the eigenvalues of the associated SOR method, λ, satisfies [11, 13]

$$(\lambda + \omega - 1) = \omega\mu\lambda^{1/2}, \tag{1.6}$$

where ω is the relaxation factor. ω_b, the optimal ω, is the determined from (1.6) to minimize $\max|\lambda|$.

If the μ are real, we have (see, e.g., [11, 13])

$$\omega_b = \frac{2}{1+\sqrt{1-\bar{\mu}^2}}, \tag{1.7}$$

where $\bar{\mu} = \max|\mu_{p,q}| = |\mu_{1,1}|$. If the μ are purely imaginary, then (see, e.g., [2, 11])

$$\omega_b = \frac{2}{1+\sqrt{1+\bar{\mu}^2}}. \tag{1.8}$$

However, when the μ are complex, the situation is not so simple. Young [11] and Young and Edison [12] have given a procedure, which is somewhat involved, for determining the optimal ω, while in a recent article Rigal [8] presents a simpler approach. In general, their efforts were directed to finding a single ω using as much as is known about the distribution of the eigenvalues of the Jacobi matrix of the original problem. We have chosen their single-eigenvalue solution and applied it to each point separately. Thus, using the formulas of Rigal [8] and letting

$$\max|\mu_{p,q}| = \mu_r + \sqrt{-1}\,\mu_i, \tag{1.9}$$

we have

$$\begin{aligned}
A &= \mu_r^2 + \mu_i^2,\\
B &= \mu_r^2 - \mu_i^2,\\
a &= A^2 - B^2,\\
b &= A^2 - B,
\end{aligned}$$

$$\begin{aligned}
\bar{\omega} = \frac{1}{A^4 - A^2 B}\,&\{[3b + (a+b^2)^{1/2}]\, a^{1/3}[(a+b^2)^{1/2} - b]^{1/3}\\
&- [3b - (a+b^2)^{1/2}]\, a^{1/3}[(a+b^2)^{1/2} + b]^{1/3}\\
&+ A^2 + 3B^2 - 4A^2 B\},\\
\omega_{ij} &= -(\bar{\omega} - (\bar{\omega}^2 + 4\bar{\omega})^{1/2})/2 \qquad \text{if} \quad A^2 > B\\
\omega_{ij} &= -(\bar{\omega} + (\bar{\omega}^2 + 4\bar{\omega})^{1/2})/2 \qquad \text{if} \quad A^2 < B.
\end{aligned} \tag{1.10}$$

It is not difficult to show that if $\mu_i = 0$, we obtain (1.7) and if $\mu_r = 0$, we obtain (1.8).

It should be pointed out that it is necessary for μ_r, the real part of max $|\mu_{p,q}|$, to be less than 1 in magnitude for convergence. If such is not the case, the method is aborted.

2. Boundary Condition Modification

Eigenvalues (1.5) were determined via (1.4) assuming given boundary conditions on all boundaries. Suppose, along one side, a normal derivative condition is given. Then (1.4a), e.g., becomes

$$a_3 v_{i-1} + a_1 v_{i+1} = \mu_1 a_0 v_i, \qquad v_0 = 0,\ v_{N-1} = v_{N+1}. \tag{2.1}$$

The largest eigenvalue in magnitude for this problem is

$$\mu_1 = \frac{2}{a_0} \sqrt{a_1 a_3} \cos \frac{\pi \alpha}{2}, \tag{2.2}$$

where α is the smallest positive zero of

$$a_3 \sin \frac{\pi \alpha (N+1)}{2} = a_1 \sin \frac{\pi \alpha (N-1)}{2} \tag{2.3}$$

if $a_1/a_3 \leqslant (N+1)/(N-1)$. If $a_1/a_3 > (N+1)/(N-1)$, then the required eigenvalue is

$$\mu_1 = \frac{2}{a_0} \sqrt{a_1 a_3} \cosh \frac{\pi \alpha}{2}, \tag{2.4}$$

where α is the positive root of

$$a_3 \sinh \frac{\pi \alpha (N+1)}{2} = a_1 \sinh \frac{\pi \alpha (N-1)}{2}. \tag{2.5}$$

Then in (1.9) we use

$$\mu_{1,1} = \mu_1 + \frac{2}{a_0} \sqrt{a_2 a_4} \cos \frac{\pi}{M+1}, \tag{2.6}$$

where μ_1 is either (2.2) or (2.4), whichever applies. If the boundary conditions are

$$v_{-1} = v_1, \qquad v_N = 0, \tag{2.7}$$

then the role of a_1 and a_3 are interchanged in Eqs. (2.3) and (2.5). Brazier [1], in his method, suggests using

$$\alpha = 1/N, \tag{2.8}$$

which is the solution of (2.3) for $a_1 = a_3$.

If the normal derivative conditions exist at both ends, i.e., the boundary conditions are

$$v_{-1} = v_1, \qquad v_{N-1} = v_{N+1}, \tag{2.9}$$

then one eigenvalue is $(a_1 + a_3)/a_0$ and the others are $(2/a_0)\sqrt{a_1 a_3}\cos(\pi k/N)$. If a_1 and a_3 have the same sign, then $\mu_1 = (a_1 + a_3)/a_0$. Otherwise it is the largest of $(a_1 + a_3)/a_0$ and $(i2/a_0)\sqrt{|a_1 a_3|}\cos(\pi/N)$. We note that Brazier [1] compromises with the use of $\alpha = 1/(2N)$ in (2.2) for the eigenvalue in this case.

In a similar manner one can handle periodic conditions. Thus suppose (1.4a) is replaced by

$$a_3 v_{i-1} + a_1 v_{i+1} = \mu a_0 v_i, \qquad v_0 = v_N,\ v_1 = v_{N+1}. \tag{2.10}$$

This leads to a circulant matrix whose eigenvalues are known, i.e.,

$$\mu_k = \left[(a_1 + a_3)\cos\frac{2\pi k}{N} + i(a_1 - a_3)\sin\frac{2\pi k}{N} \right] \Big/ a_0, \qquad k = 1,\dots, N. \tag{2.11}$$

If a_1 and a_3 have the same sign, then the largest eigenvalue in magnitude is $(a_1 + a_3)/a_0$ for $k = N$. When a_1 and a_3 differ in sign, the largest eigenvalue in magnitude occurs when k is that integer which is closest to $N/4$. In any event.

$$\mu_{1,k} = \frac{1}{a_0}\left[2\sqrt{a_2 a_4}\cos\frac{\pi}{M+1} + (a_1 + a_3)\cos\frac{2\pi k}{N} + i(a_1 - a_3)\sin\frac{2\pi k}{N} \right] \tag{2.12}$$

is used in (1.9). (Brazier does not consider periodic conditions.)

3. Modifications Due to Region Shape

In each instant where the eigenvalue was to be determined, it was necessary to use the number of unknowns along each axis. In a rectangle, there is no ambiguity. But

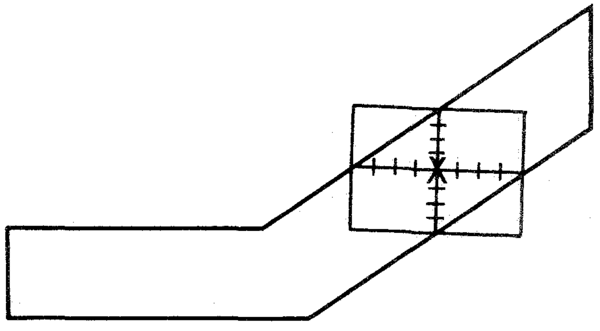


FIG. 1. Selection of grid size in eigenvalue selection.

what if the region is not a rectangle? Experimental results indicate the following: At a given point, the value of M and N that should be used is that number of unknowns between the boundaries along a horizontal and vertical line through the point. In Fig. 1, we show the enclosing rectangle to use at an arbitrary point. This idea was also suggested by Brazier [1].

4. Some Numerical Results

We applied the technique presented in this paper to several different problems, comparing the results with those of the SOR method. The problems chosen were based on the experience of the author. In all cases, the initial guess was identically zero and iteration proceeded until the maximum change in any component was less than 10^{-6}. These problems and results are as follows:

Problem I. Poisson's equation in an L-shaped region (Fig. 2a). Table I contains some results. It seems clear that for this problem the ad hoc SOR is as effective as the SOR at optimal ω.

Problem II. Poisson's equation in a quarter circle or quarter ellipse using short-leg difference approximations (Fig. 2b).

Table II contains results for this problem. Again, we see that the convergence rate of the ad hoc SOR method is comparable to that of SOR at optimal ω. However, the choice of ω for SOR can be critical. For example, in Table IIa, for the quarter circle with $N=63$ and $\omega_b=1.895$, 118 iterations were required. For $\omega=1.89$, 130 iterations were necessary while $\omega=1.88$ led to over 150 iterations.

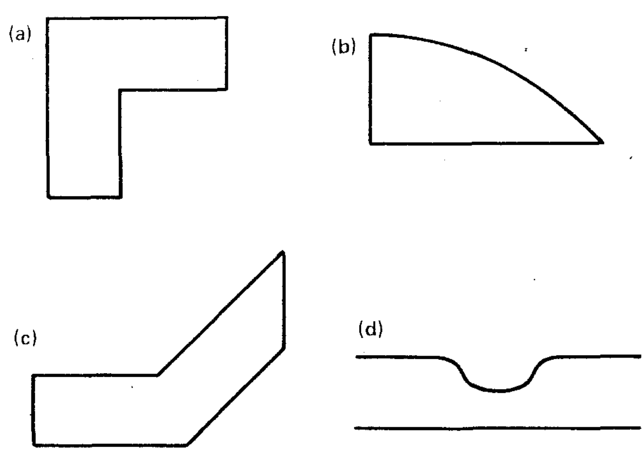
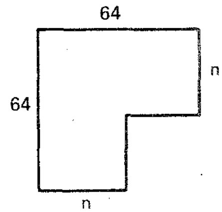


FIG. 2. Region shapes.

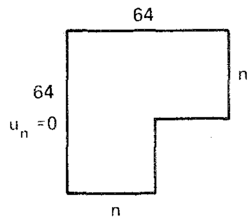
TABLE I

Ia)



n	AD-HOC SOR	SOR	(ω_b)
15	44(1.91-1.72)*	44	(1.78)
31	93(1.91-1.85)	94	(1.88)
47	114(1.91-1.89)	116	(1.91)

Ib)



n	AD-HOC SOR	SOR	(ω_b)
15	80(1.93-1.73)	81	(1.86)
31	118(1.93-1.86)	125	(1.915)
47	153(1.93-1.90)	159	(1.93)

*RANGE OF ω's COMPUTED

PROBLEM III. The calculation of the vorticity in a fluid flow problem in a symmetric Y-shaped region (Fig. 2c).

This problem is described in detail in [3]. The equation is of the form

$$\nabla^2 u = Re(a(x, y)\, u_x + b(x, y)\, u_y), \tag{4.1}$$

where a and b are functions of x and y and Re is a positive constant (usually the Reynolds number). The difference equations has the stencil

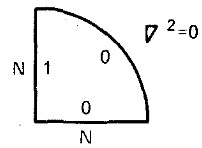
$$\begin{bmatrix} & 1 - Re b_{ij} & \\ 1 + Re a_{ij} & -4 & 1 - Re a_{ij} \\ & 1 + Re b_{ij} & \end{bmatrix}, \tag{4.2}$$

where a_{ij} and b_{ij} vary over the grid.

The results for this problem are shown in Table III. Here we begin to see how effective the ad hoc SOR method can be. For those linear systems which require severe underelaxation, the ad hoc method yields dramatic convergence. Further evidence of this appears in the next problem.

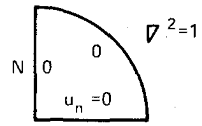
TABLE II

a)



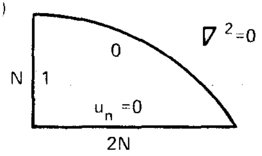
N	AD-HOC SOR	SOR (ω_b)
31	59(1.82-1.06)	61(1.80)
63	115(1.91-1.03)	118(1.895)

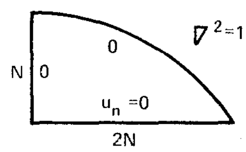
b)



15	39(1.74-1.22)	39(1.72)
31	77(1.86-1.07)	68(1.851)
63	149(1.92-1.03)	146(1.920)

c)





N	AD-HOC SOR	SOR	AD-HOC SOR	SOR
31	112(1.91-1.02)	129(1.894)	120(1.91-1.02)	107(1.900)

Problem IV. The calculation of the stream function in a fluid flow problem involving a stenosised tube (Fig. 2d) analytically mapped into a rectangle in axisymmetric coordinates (see [4] for details).

The original equation is of the form

$$u_{zz}+u_{rr}-u_r/r=-rh(r,z), \tag{4.3}$$

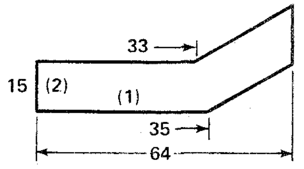
where $h(r,z)$ is some function and after mapping becomes

$$(u_{xx}+u_{yy})(x_z^2+x_r^2)-u_x x_r/r-u_y x_z/r=-rh(x,y), \tag{4.4}$$

where r is a function of x and y. This leads to a stencil of the form

$$\begin{bmatrix} & C-\alpha g_{ij} & \\ 1+\alpha f_{ij} & -2(C+1) & 1-\alpha f_{ij} \\ & C+\alpha g_{ij} & \end{bmatrix}, \tag{4.5}$$

TABLE III



a) DIRICHLET PROB.

Re	AD-HOC SOR	SOR (ω_b)	BRAZ
0	84(1.77-1.07)	85(1.71)	91
5	55(1.75-1.07)	56(1.65)	63
50	34(1.61-0.65)	103(0.73)	D*
200	90(1.03-0.22)	>200(0.097)	D
b) $u_n = 0$ on (1)			
0	107(1.86-1.07)	125(1.81)	
5	62(1.84-1.07)	64(1.72)	72
10	44(1.83-1.07)	49(1.57)	D
50	37(1.65-0.65)	103(0.73)	D
c) $u_n = 0$ on (2)			
0	89(1.78-1.07)		91
5	65(1.78-1.07)		88
10	84(1.78-1.07)	106(1.69)	129

*D = DIVERGENCE

where f_{ij} and g_{ij} are grid functions, C is a constant related to the mesh ratio, and α was inserted in the difference equation to allow a parameter study.

Table IV contains some results. We again see the effectiveness of the ad hoc SOR over the SOR for large α. Indeed, for such systems, ω_b is very near the range of divergent ω's. Thus, for Problem IVb, ω_b for $\alpha = 25$ is 0.39. However, an ω of 0.42 caused divergence. It would appear that SOR is not the method to use for stencils of the form (4.2) and (4.5).

PROBLEM V. An equation of the form

$$u + W(a(x, y)\, u_y + b(x, y)\, u_x) = h(x, y) \tag{4.6}$$

is an L-shaped region.

This type of equation was encountered in the solution of a flow problem involving a fluid of second grade described in [5]. The difference stencil has the form

$$\begin{bmatrix} & Wf_{ij} & \\ -Wg_{ij} & -d_{ij} & Wg_{ij} \\ & -Wf_{ij} & \end{bmatrix}. \tag{4.7}$$

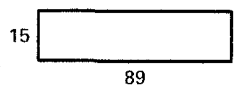
Indeed, the techniques of this paper were devised to solve this problem [5]. Some results are presented in Table V. Although the convergence is not as dramatic as that in Problems III and IV, note that for $W = 0.5$, ω_b was 0.076. For a change of as little as 0.001, i.e., $\omega = 0.077$, the SOR method diverged. Thus it is almost impossible to get an estimated ω to give reasonable convergence without running the problem for a series of ω's. The ad hoc SOR spares us this agony.

PROBLEM VI. The Poisson equation in a rectangle using logarithmic varying mesh in each direction.

The results for this problem are shown in Table VI. The ad hoc SOR method did not appear to be as effective as the SOR for certain cases. To investigate this more closely, it was decided to run a one-dimensional counterpart to this problem. The exact theoretical ω_b for the one-dimensional problem was determined from the

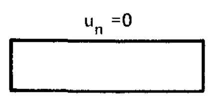
TABLE IV

IVa)



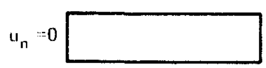
a	AD-HOC SOR	SOR (ω_b)
1	43(1.72-1.40)	43(1.70)
50	41(0.93-0.084)	103(0.19)

IVb)



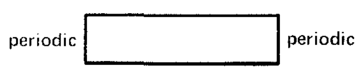
a	AD-HOC SOR	SOR (ω_b)
1	74(1.82-1.39)	68(1.78)
25	26(1.3-0.16)	93(0.39)
50	41(0.93-0.082)	

IVc)



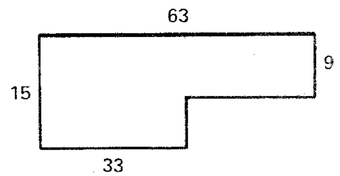
a	AD-HOC SOR	SOR (ω_b)
1	44(1.72-1.39)	
25	28(1.3-0.16)	
50	41(0.93-0.084)	

IVd)



a	AD-HOC SOR	SOR (ω_b)
1	43(1.72-1.38)	43(1.70)
50	43(0.93-0.084)	106(0.19)

TABLE V



W	AD-HOC SOR	SOR (ω_b)
0.1	65(0.993-0.223)	78(0.32)
0.5	206(0.86-0.05)	256(0.076)

spectral radius of the Jacobi matrix which was computed using EISPACK. The results are shown in Table VII.

There apparently are two ω_b's—one theoretical and one computational. The theoretical ω_b depends on the difference equation, the boundary conditions, and the region shape. The computational ω_b depends on these, and also the right-hand side, the initial guess, the norm used in the computation, the convergence criterion, etc. In most cases, these agree or differ negligibly. But occasionally, the difference can be marked and the number of iterations can be quite different. Table VII displays such a situation. Also we note that the two problems in Table IIc have the same coefficient matrix but different right-hand sides. They have the same theoretical ω_b but, as the table shows, each has its own computational ω_b.

5. Other Variable Relaxation Techniques

To our knowledge, there are three other variable relaxation factor techniques in the literature. We briefly comment on each.

(a) *Brazier* [1]

This approach appears to be closest to our technique. Based on stencil (1.2), the equations are as follows: Let

$$\begin{aligned} \overline{a_{13}} &= \max(a_1, a_3), \\ \underline{a_{13}} &= \min(a_1, a_3), \\ \overline{a_{24}} &= \max(a_2, a_4), \\ \underline{a_{24}} &= \min(a_2, a_4). \end{aligned} \tag{5.1}$$

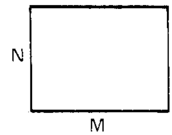
Then

$$a = a_0 - \underline{a_{13}} \cos(\pi/N) - \underline{a_{24}} \cos(\pi/M),$$

$$b = \underline{a_{13}} \sin(\pi/N) + \underline{a_{24}} \sin(\pi/M),$$

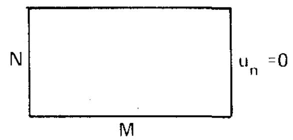
TABLE VI

a)



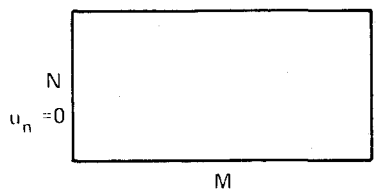
N	M	AD-HOC SOR	SOR (ω_b)	BRAZ
15	15	32(1.673-1.671)	29(1.69)	29(1.70-1.69)
31	31	60(1.821-1.819)	55(1.84)	54(1.84-1.83)
63	63	111(1.906-1.905)	98(1.916)	101(1.92-1.91)
15	63	98(1.904-1.881)		87(1.91-1.90)
63	15	64(1.861-1.741)		81(1.90-1.78)

b)



15	15	49(1.823-1.759)	47(1.810)	49(1.83-1.78)
31	31	98(1.892-1.856)	92(1.902)	87(1.91-1.88)
63	63	187(1.944-1.925)	190(1.960)	164(1.96-1.94)
15	63	175(1.943-1.906)	177(1.956)	175(1.96-1.94)
63	15	88(1.890-1.840)	85(1.892)	87(1.91-1.87)

c)



N	M	AD-HOC SOR	SOR (ω_b)	BRAZ
15	15	45(1.82-1.76)	43(1.77)	49(1.83-1.76)
31	31	89(1.89-1.86)	76(1.883)	92(1.91-1.88)
63	63	172(1.94-1.93)	134(1.940)	160(1.96-1.94)
15	63	151(1.94-1.91)	119(1.934)	164(1.96-1.94)
63	15	85(1.89-1.84)	81(1.876)	93(1.91-1.87)

TABLE VII

	N	AD-HOC SOR	SOR (ω_b–theor.)	SOR (ω_b–comp.)
(a)				
	7	18 (1.44-1.44)	18 (1.449)	16 (1.46)
	15	34 (1.67-1.67)	33 (1.675)	31 (1.70)
	31	64 (1.82-1.82)	63 (1.822)	56 (1.83)
	63	119 (1.90-1.90)	118 (1.907)	94 (1.914)
(b)			$u_n = 0$	
	7	32 (1.72-1.70)	35 (1.672)	31 (1.71)
	15	59 (1.85-1.84)	70 (1.833)	59 (1.84)
	31	129 (1.92-1.91)	135 (1.916)	117 (1.920)
	63	261 (1.96-1.95)	252 (1.958)	212 (1.960)
(c)		$u_n = 0$		
	7	27 (1.62-1.64)	28 (1.603)	26 (1.62)
	15	48 (1.79-1.80)	57 (1.789)	52 (1.80)
	31	107 (1.89-1.89)	111 (1.891)	81 (1.898)
	63	212 (1.94-1.94)	207 (1.945)	165 (1.948)

$$
\begin{aligned}
c &= \overline{a_{13}} \cos(\pi/N) + \overline{a_{24}} \cos(\pi/M), \\
d &= \overline{a_{13}} \sin(\pi/N) + \overline{a_{24}} \sin(\pi/M), \\
f &= 2(a-c)/(a^2 - c^2 + b^2 - d^2), \\
g &= (ad^2 - cb^2)/(a-c) - ac, \\
\omega_{ij} &= fa_0/(1 + \sqrt{1 + f^2 g}).
\end{aligned}
\tag{5.2}
$$

The arguments of the trigonometric functions are appropriately modified when derivative boundary conditions are used, as described in Section 2.

The method appears to be most effective for matrices whose off-diagonal elements all have the same sign. Considering our problems, we have the following: For Problems I and II, this method is as effective as both ad hoc SOR and SOR. For Problem III, Table III contains some results indicating divergence usually occurs when some of the off-diagonal coefficients change sign. For Problem VI, the method produces results similar to those of the ad hoc SOR, subject to numerical fluctuations.

In Brazier's paper [1] he applied his method to an L-shaped region containing a rectangular hole. Using a variety of boundary conditions on the region, his application included both a "uniform coefficient set" and a "nonuniform coefficient set." We applied the ad hoc SOR (and Brazier's method) to the latter and duplicated his results. Thus generally, the ad hoc SOR method is as effective as Brazier's method when the latter converges and appears to converge for some problems for which Brazier's method does not.

(b) *Russel* [9]

This method was devised specifically for equations whose stencil have the form

$$\begin{bmatrix} & 1-b & \\ 1+a & -4 & 1-a \\ & 1+b & \end{bmatrix}. \tag{5.3}$$

The formulas is

$$\omega_{ij} = 2/(1 + \sqrt{\tfrac{1}{2}[(a^2 + b^2) + \pi^2(N^{-2} + M^{-2})]}), \tag{5.4}$$

which was determined empirically. The method produced results corresponding to those of the ad hoc SOR and the SOR for Problems I, II, and III, problems for which it was devised. However, the technique deteriorated or was not applicable to Problems IV, V, and VI.

(c) *Strikwerda* [10]

Considering the stencil of (1.2), Strikwerda's formulas formula is

$$\omega_{ij} = 2\bigg/\left(a_0 + \sqrt{a_0\left(\frac{(a_1 - a_3)^2}{(a_1 + a_3)} + \frac{(a_2 - a_4)^2}{(a_2 + a_4)}\right)}\right). \tag{5.5}$$

Although this appears to be applicable to general difference equations, it was developed specifically for stencils of the form (5.3) for very large a and b of order 10–100,000. As it turned out, this approach was not competitive with ad hoc SOR for the problems considered here, but may well be, however, for the problems for which it was designed. We further note that Strikwerda has some analytical results [10] and also relaxation factors for nine-point stencils.

6. Comments and Conclusions

The computation of the ω's will require some extra work, equivalent to several SOR iterations. However, if we have no idea what the optimal ω might be, this extra work can be easily recovered in producing a near optimal iteration procedure, especially for problems of type III and IV, where the optimal ω is dangerously close to the divergent region (Figs. 3, 4 in [8]).

For those problems with, e.g., normal derivative conditions, the solution of Eqs. (2.3) or (2.5) can increase the time needed for determining the ad hoc ω's. If only a few solutions are desired, it may well be sufficient to use Brazier's choice (2.8) to save some time without seriously retarding convergence.

There is always a danger in making generalizations based on limited experience. Nonetheless, our results here are so tempting, we are inclined to recommend the ad hoc SOR, at least for problems that are to be solved relatively few times. In particular, problems of type III and IV should use the ad hoc method (type III can also use Russel's). Problems of type V appear to have no more effective iterative approach than ad hoc SOR. Clearly, however, more investigation is needed.

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